Critical points in layered systems

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Critical points that do not involve any symmetry change are shown to define a set of new universality classes in layered systems. Extension of this problem to higher dimensions may be made in an infinite number of ways which define continuously varying upper critical dimensions. This allows us to estimate uncertainties on the critical-exponent values obtained in a first-order Wilson expansion. The corresponding physical situations include the electric field induced chiral-smectic-C–chiral-smectic-C critical point, the smectic-A–smectic-A critical point, but also the smectic-C–smectic-I and smectic-C–smectic-F critical points which have not been discovered experimentally yet. Recent experiments provide encouraging support to the current analysis.

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I. INTRODUCTION

Phase transitions that do not involve symmetry change are generally discontinuous. In a pressure-temperature plane for a one-component system, they define a line of first-order transitions, which may, however, terminate on a point where all discontinuities vanish. This is the well-known case of the liquid-vapor critical point [1]. The phase-separation phenomenon in a binary mixture defines a surface of a first-order transition in a chemical potential's (of species one and two) temperature three-dimensional space, which terminates on a line of consolute points where again all discontinuities vanish [2]. Similar phenomena may occur in more complex systems such as microemulsions [3] or nematic liquid-crystal mixtures [4]. In all these cases the behavior of the system is governed by the Ising universality class in the vicinity of the critical point [5], although corrections to scaling may be system dependent and nontrivial.

The situation is different in systems for which the long-wavelength properties are described by a displacement variable. For instance, in gels, the existence of a connected network suppresses the concentration fluctuations characteristic of a consolute point and forces a mean-field behavior [6,7].

Similar phenomena occur in crystalline ferroelectrics [8,9]; layered systems (i.e., smectic liquid crystals) are more original and define new universality classes. This has been recognized already in the case of the Sm-A–Sm-A (Sm-A denotes smectic-A phase) critical point [10], and is generalized in this article. The originality results from the existence of qualitatively different spacial directions as discussed in the following. Although our calculations apply for the Sm-A–Sm-A critical point in the absence of any symmetry breaking field or in the presence of a transverse magnetic (electric) field for negative magnetic (dielectric) anisotropy, we focus our attention mainly on the Sm-C^*–Sm-C^* critical point to be described below for which experimentation should be easier [11].

It is worth stressing that the Sm-A–Sm-A critical point can correspond to either an ordinary consolute point in a one-dimensional (1D) ordered matrix [12] or the termination of a layer thickness discontinuity line resulting from incommensurability [13] (the only difference bearing on the critical domain). As already stressed, consolute points in isotropic liquids belong to the same universality class as the liquid-vapor critical point (i.e., Ising). One could thus, at first sight, expect the Sm-A–Sm-A critical point to belong also to this class. The coupling between layer spacing and concentration fluctuations in fact generates interactions which have no counterparts in either the liquid-vapor or the ordinary consolute point cases. Superfluids, which can also be described by a phase variable [14], do not exhibit the same type of coupling because of both time reversal and spatial symmetry. As a result, consolute points in superfluids should simply belong to the compressible Ising universality class. Another point worth stressing concerns the Sm-C–Sm-I (Sm-C–Sm-F) transition: Sm-C are stacks of fluid layers in which molecules are cooperatively tilted with respect to the stacking direction. Sm-I (Sm-F) obey the same definition, but differ from Sm-C by the existence of a hexatic bond order. This bond order, however, does not change the point group symmetry, and as a result there can be either a first-order transition or no transition at all between phases recognized as Sm-C and Sm-I (Sm-I) [15–17]. The existence of Sm-C–Sm-I first-order transitions for some systems, together with the absence of any transition for others, implies the existence of an isolated critical point in a suitable phase diagram. This point should belong to the same universality class as that of a consolute point in a Sm-C matrix. We show in Appendix A that it is in fact identical to that defined by the Sm-A–Sm-A universality class, and hence relevant to the current analysis.

When Sm-C are made of chiral molecules (labeled Sm-C^*), they become ferroelectric, the macroscopic average polarization being both perpendicular to the stacking and the tilt directions [18,19]. The chirality is also responsi-

ble for a helical precession of the polarization direction, which a modest externally applied electric field can unwind. In all that follows, fields higher than the unwinding threshold will be present so that we will not have to worry about this precession. This would not be the case for a Sm-C^* consolute point.

Sm-C differ from Sm-A through the existence of the tilt which can grow continuously from zero: as a result the Sm-C–Sm-A transition is in general second order [20]. Of couse, this does not preclude the existence of first-order transitions as found in materials with a strong polarization [21]. When an electric field is applied parallel to the layers in the Sm-A phase (made of chiral molecules), both polarization and tilt are induced via the electroclinic effect [22], and there is no longer any symmetry difference between this phase and the low-temperature Sm-C^*. The situation is then comparable to that existing in piezoelectric materials, in an (E,T) plane one expects a first-order line terminating on a Sm-C^*–Sm-C^* critical point. A Landau expansion in terms of the spontaneous polarization and the external field only, accounts qualitatively satisfactorily for the experimentally observed behavior [23]. In view of our general understanding of phase transitions, it is important to find the universality class this point belongs to: we show that it is in fact original.

If B is the layer compression elastic modulus, $\delta\epsilon$ and δC_P are the critical part of the electric permittivity and the specific heat, respectively, and χ_θ is the tilt susceptibility, we expect

$$B^{-1} \propto \delta\epsilon \propto \delta C_P \propto \chi_\theta = t^{-\gamma} f(\delta h t^{-\Delta}) , \tag{1}$$

where t and δh are scaling fields linearly related to $(T-T_C)$ and $(E-E_C)$. The exponents γ and Δ (which have their standard meaning [24]) are calculated to first order in a two-dimensional ϵ expansion to be defined in Secs. III and IV. Although our values of γ and Δ are clearly poorly defined, their ratio γ/Δ, which should be directly accessible to experiment in a straightforward approach of the critical point, is expected to be comprised between 0.375 and 0.395 for Sm-C^*–Sm-C^* (between 0.31 and 0.34 for Sm-A–Sm-A). $\delta\epsilon$ and χ_θ have already been measured, and there are indications that close to the critical point γ is smaller than 1, as suggested by our first-order expansion [25]. δC_P should be measured with the techniques developed for thin oriented samples [26]. Light scattering in the planar geometry [27] should be ideal for studying the vanishing of B, but also the divergence of the curvature elastic moduli. We expect the scattering to be controlled by the ratio

$$I(q) \propto \frac{k_B T q_z^2}{B_z q_z^2 + B_y q_y^2 + K_{xx} q_x^4 + K_{yy} q_y^4 + K_{zz} q_z^4 + K_{zx} q_z^2 q_x^2 + K_{zy} q_z^2 q_y^2 + K_{xy} q_x^2 q_y^2} \tag{2}$$

with

$$B_z = t^{\gamma} f(\delta h t^{-\Delta}), \quad B_y = \text{const} , \tag{3a}$$

$$\xi_z^2 = \frac{K_{zz}}{B_z} \propto t^{-2\nu_z} f_z(\delta h t^{-\Delta}) , \tag{3b}$$

$$\xi_y^2 = \frac{K_{yy}}{B_y} \propto t^{-2\nu_y} f_y(\delta h t^{-\Delta}) , \tag{3c}$$

$$\xi_x^4/\xi_z^2 = \frac{K_{xx}}{B_x} \propto t^{-4\nu_x + 2\nu_z} f_x(\delta h t^{-\nabla}) , \tag{3d}$$

with ξ_z, ξ_y, and ξ_x the correlation lengths in the z, y, and x directions, respectively, and ν_z, ν_y, and ν_x the corresponding exponent, different from each other to first order in the expansion scheme.

In a single experiment one could, in principle, be able to measure γ, Δ, ν_z, ν_y, and ν_x by monitoring T and E and varying wave vectors. Measurements of the tilt angle, the layer spacing, and the polarization should provide essentially β/Δ in the vicinity of the critical point, with the proviso that (as in the earlier work on the Sm-A–Sm-A critical point) we have not calculated explicitly the equation of state and the shape of the coexistence curve. We expect the law of rectilinear diameters not to hold, but diameters should rather follow a law $\delta h \propto t^{\Delta}$. The only critical exponent that is not easily accessible to static experiments is the specific-heat exponent α at constant layer spacing: ultrasound attenuation should, however, have access to it.

The study of the x-ray line shape may be less instructive: because of the existence of the B_y term in (2), Sm-C^*, in the presence of the external electric field, should exhibit true long-range order, and thus show Bragg peaks. At the critical point, in the Gaussian approximation, one should recover a Caillé type of singularity [28], governed by B_y rather than B_z. However, since the η exponents are positive, this result certainly does not survive when the Gaussian approximation is lifted, and Bragg peaks should survive even at the critical point. Eventually the scaling laws (3) are expected to hold irrespective of the expansion scheme.

In the second section we recall briefly the mean-field analysis of the Sm-C^*–Sm-C^* critical point, in terms of polarization, discuss the importance of other coupled variables such as tilt and layer spacing and eventually discuss the relevance of the choice of the layer displacement variable as order parameter. This shows the similarity of this problem with the Sm-A–Sm-A critical point in the presence of a (strong) transverse electric- (magnetic-) field [and negative dielectric (magnetic) anisotropy]. The similarities and differences with crystalline ferroelectrics are also discussed [8,9].

In Sec. III we define an anisotropic extension scheme

to higher dimensions which allows us to include the zero field Sm-A–Sm-A problem in our considerations (and hence Sm-C–Sm-I and Sm-C–Sm-F critical point), and vary continuously the upper critical dimension.

Section IV is devoted to the derivation of the renormalization-group recursion relations and the search for fixed points, whereas eigenvalues and critical exponents are calculated in Sec. V.

In Sec. VI we discuss the meaning of our results and recall the strongest predictions provided by our analysis.

II. MEAN-FIELD ANALYSIS AND VARIABLES CHOICE

Let us start with a simple Landau expansion in terms of the total polarization P of the Sm-C^* phase:

$$F-F(P=0)=\int d^3x\left[\sum_{n=2}^{\infty}\frac{f_{2n}}{(2n)!}p^{2n}-PE+\tfrac{1}{2}(\nabla P)^2\right] . \tag{4}$$

In this expression both the macroscopic electric fields E and P are taken in the y direction as defined in Fig. 1, and suitable rescaling is assumed to impose the value of $\frac{1}{2}$ for the gradient term coefficient in all directions. In Eq. (4) all other variables characteristic of the system are considered as relaxed to their equilibrium value for the given P, but the layers are kept flat. The coupling constants f_{2n} are functions of temperature and pressure. In the following, without loss of generality, we keep the temperature dependence only. We further consider the case when $f_4<0$ and f_2 changes sign at a temperature T_0, which ensures that the transition to a ferroelectric state is first order for $E=0$. As E is increased, the discontinuity in P decreases and eventually vanishes at a critical point (T_c,E_c) (Fig. 2). Right at this point, the lowest term of the free-energy expansion as a function of $p=P-P_c$ is quartic:

$$F-F(P_c,E_c)=\int d^3x\left[\sum_{n=4}\frac{\tilde{f}_n}{n!}p^n+\tfrac{1}{2}(\nabla p)^2\right] . \tag{5}$$

The compatibility with Eq. (4) defines three relations for the three parameters P_c, E_c, and T_c:

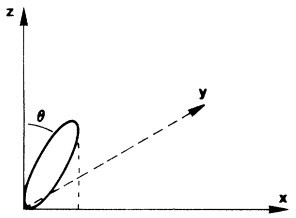

FIG. 1. Schematic representation of the axis conventions used throughout the paper: z is the layers normal, x the tilt direction, y the average polarization and external electric-field direction.

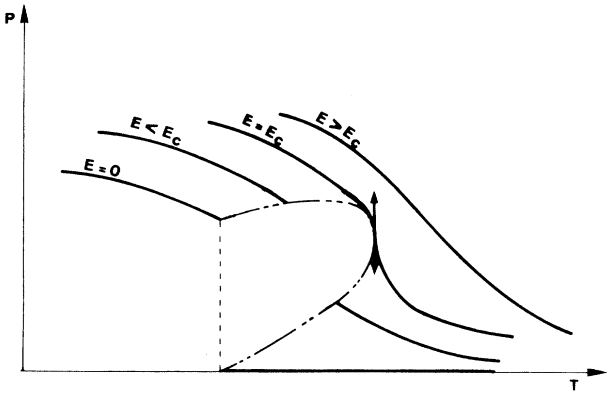

FIG. 2. Schematic phase diagram in a polarization-temperature plane, including an isolated Sm-C^*–Sm-C^* critical point (11). Solid lines correspond to $P(T)$ curves at constant external field. Dashed lines show the first-order discontinuities. The dash-dotted curve indicates the "coexistence domain." For $E<E_c$, $P(T)$ curves are discontinuous, for $E\geq E_c$ they are continuous.

$$\begin{aligned}
\tilde{f}_1^c&=\sum_{n=1}^{\infty}\frac{p^{2n-1}}{(2n-1)!}f_{2n}^c-E_c=0 ,\\
\tilde{f}_2^c&=\sum_{n=1}^{\infty}\frac{p^{n-2}}{(2n-2)!}f_{2n}^c=0 ,\\
\tilde{f}_3^c&=\sum_{n=2}^{\infty}\frac{p_c^{2n-3}}{(2n-3)!}f_{2n}^c=0 ,
\end{aligned} \tag{6}$$

where $f_{2n}^c=f_{2n}(T_c)$. With the usual assumption $f_2=\alpha(T-T_0)$, f_{2n} constant, and keeping terms up to sixth order only one finds

$$\begin{aligned}
E_C&=\frac{4\sqrt{6}}{5}\frac{(-f_4)^{5/2}}{(f_6)^{3/2}} ,\\
P_C&=\left[-\frac{6f_4}{f_6}\right]^{3/2} ,\\
T_C&=T_0+\frac{3}{2}\frac{f_4^2}{f_6}\frac{1}{\alpha} .
\end{aligned} \tag{7}$$

Within this kind of description, the problem is identical to that of standard ferroelectrics or the liquid-vapor case. The differences comes from the couplings with other degrees of freedom: the tilt angle θ and the layer thickness a.

Although θ is essential in the existence of the Sm-A^*–Sm-C^* transition (P is, in fact, a secondary order parameter for this transition), it is not essential for determining the universality class that the Sm-C^*–Sm-C^* critical point belongs to. This is so because the free-energy functional has the same formal expansion whether it is expressed in terms of P or θ. For instance, the free energy used in Eq. (1) of Ref. [29] can be cast in the form of Eq. (4) with either θ or P as the variable provided a first minimization is performed on P or θ, respectively.

The coupling to the layer thickness a is more subtle: in a strict mean-field analysis one can write (with obvious

notations)

$$(P-P_c)\propto(\theta-\theta_c)\propto(a-a_c)\ , \tag{8}$$

and the expansion around the critical point can be performed in any of these variables. However, if one wants to keep track of the gradient terms and rotational invariance, the situation is different. One should realize that the fluctuating variable is the layer displacement rather than the layer thickness. As a consequence, nonanalytic terms are generated both in the harmonic and anharmonic parts of the free-energy expansion if one chooses p as the variable. For instance, at the harmonic level, the coupling of p to the smectic elastic degrees of freedom reads

$$F_c=\int d^3x\left[CP\nabla_z u+\frac{B_z}{2}(\nabla_z u)^2+\frac{B_y}{2}(\nabla_y u)^2+K(\Delta_{xx}u)^2\right]\ , \tag{9}$$

where

$$\nabla_z u=\frac{a-a_c}{a_c}+O\left[\left(\frac{a-a_c}{a_c}\right)^2\right]\ , \tag{10}$$

in which C expresses the coupling between P and the layer thickness, B_z is the compressional elastic modulus at constant polarization, B_y expresses the energy cost of tilting the layers away from the y direction ($B_y=P_cE_c$), and K is the standard smectic curvature modulus. Minimizing Eq. (9) with respect to u yields a redefinition of $\tilde{f}_2$, which reads, in q space,

$$\tilde{f}_2^{\rm eff}=\tilde{f}_2-\frac{c^2q_z^2}{B_zq_z^2+B_yq_y^2+K_1q_x^4}\ . \tag{11}$$

$\tilde{f}_2^{\rm eff}$ is both nonanalytic and highly anisotropic.

Furthermore, symmetry allows (and rotational invariance requires) the existence of coupling terms of the type

$$F_c'=w\int d^3x\,p(\nabla_x u)^2\ ,$$

which after minimization with respect to u yields nonanalytic terms such as

$$F''=f_3'\int d^3x\,p(\nabla_z^{-1}\nabla_x p)^2\ .$$

F'' is globally cubic in p, but qualitatively different from the term described by $\tilde{f}_3$. In particular it is not possible to cancel simultaneously $\tilde{f}_3$ and f_3', which drives this problem away from the analogy with the liquid-vapor case. In view of the awkwardness of expressions (10) and (12), we choose to describe the Sm-C^*–Sm-C^* critical point in terms of the displacement variable u. The general form of the free energy can be obtained from symmetry considerations: (i) Because of translational invariance, only gradients of u can enter the expansion. (ii) The twofold symmetry along the y axis imposes the expression to be invariant in the simultaneous sign change of z, x, and y. (iii) Rotational invariance in the (z,x) plane imposes the combination $\{\nabla_z u+\frac{1}{2}[(\nabla_x u)^2+(\nabla_z u)^2]\}=E(u)$ to enter the free energy as a unique variable, which yields Ward identities similar to that relevant to the Sm-A–Sm-A problem (10). One thus obtains the free energy

$$\begin{aligned}H_1(u)=\int d^3x\Big\{&h\nabla_z u+\tfrac{1}{2}[B_z(\nabla_z u)^2+B_y(\nabla_y u)^2+B_x(\nabla_x u)^2]\\&+\tfrac{1}{2}[K_{xx}(\nabla_{xx}^2u)^2+K_{yy}(\nabla_{yy}^2u)^2+K_{zz}(\nabla_{zz}^2u)^2+K_{xz}(\nabla_{zx}^2u)^2+K_{yx}(\nabla_{xy}^2u)^2+K_{zy}(\nabla_{yz}^2u)^2]\\&+\frac{W_1}{3!}(\nabla_z u)^3+\frac{W_2}{2}\nabla_z u(\nabla_x u)^2+\frac{W_2'}{2}\nabla_z u(\nabla_y u)^2+\frac{V_1}{4!}(\nabla_z u)^4+\frac{V_2}{4!}(\nabla_x u)^4+\frac{V_2'}{4!}(\nabla_y u)^4\\&+\frac{V_{12}}{4}(\nabla_z u)^2(\nabla_x u)^2+\frac{V_{12}'}{4}(\nabla_z u)^2(\nabla_y u)^2+V_{12}''(\nabla_x u)^2(\nabla_y u)^2+\cdots\Big\}\ ,\end{aligned} \tag{12}$$

where the ellipsis represents higher-order terms, with the relations required by rotational invariance in the xz plane:

$$h=B_x,\quad B_z=h+W_2,\quad V_2=3W_2,\quad V_{12}=W_1-2W_2\ . \tag{13}$$

In $H_1(u)$, θ and P are assumed to have reached their local equilibrium in the presence of the layer distortion. Furthermore, $H_1(u)$ is the part of the free energy which arises from short-range interactions; one should add to it the long-range part which comes from Coulomb interactions:

$$H_2=\int d^3x\int d^3x'[\nabla\cdot\mathbf{P}(\mathbf{x})\nabla\cdot\mathbf{P}(\mathbf{x}')]/|\mathbf{x}-\mathbf{x}'|)\ , \tag{14}$$

which can be rewritten

$$H_2=K\int d^3x\int d^3x'\{[(\nabla_{z'}\nabla_{y'}u(x')\nabla_z\nabla_y u(\mathbf{x})]/|\mathbf{x}-\mathbf{x}'|\}\ . \tag{15}$$

If we take as the reference thickness a_c, the critical point is defined by $h^c=B_z^c=W_1^c=0$, in a mean-field description. [Note that relation (13) implies $B_x^c=W_2^c=V_2^c=V_{12}^c=0$.] Indeed, from relations (8,10) we know that if we ignore the tilt and curvature fluctuations of the layers,

expression (12) must reduce to expression (5) at the critical point (with the replacement of p by $\nabla_z u$). This implies that the smectic compressional elastic constant must vanish at the critical point. Formally the problem is identical to that of a Sm-A–Sm-A critical point in a system with a negative magnetic (or dielectric) anisotropy and submitted to a magnetic (or ac electric) field in the y direction, if one neglects H_2. Since we will show (at least to first order in the expansion scheme defined in the following) that H_2 is irrelevant in the renormalization-group sense, we expect that these two problems belong to the same universality class. Their key characteristic is a propagator $G(q)$ involving different types of elasticities in different directions $G^{-1}(q) = (B_z q_z^2 + B_y q_y^2 + K_{xx} q_x^4 + K_{yy} q_y^4 + K_{zz} q_z^4 + K_{xz} q_z^2 q_x^2 + K_{yx} q_x^2 q_y^2 + K_{zy} q_y^2 q_z^2)$ involving no $B_x q_x^2$ along the line of maximum susceptibility, and with $B_x^c = 0$, $B_y^c \neq 0$, and the existence of at least two qualitatively different cubic terms.

There is also some resemblance with isolated critical points in the (T,E) plane of crystalline ferroelectrics like KH_2PO_4 (KDP) (9) and more generally with structural phase transitions (8). However, whereas the coupling to strain generally decreases the upper critical dimension in crystalline systems, it does not decrease in layered ones if there is at least one soft direction (i.e., x, in which only curvature plays a role). Note that when there is no soft direction in smectic liquid crystals (e.g., Sm-A–Sm-A critical point with positive diamagnetic anisotropy in presence of a magnetic field along the symmetry axis) the upper critical dimension is also reduced below 3 [30]. We will show, in Sec. III, that the problem at hand is nontrivial and can be extended to higher dimensions in an infinite number of ways.

III. EXTENSION TO HIGHER DIMENSIONS AND UPPER CRITICAL DIMENSIONS

In conventional critical points, all directions are equivalent, and extending the problem to a higher dimension is straightforward. In this problem, there is some flexibility since one can *a priori* add either x-, y-, or z-like dimensions, which all are qualitatively distinct. We choose to add x-like and y-like dimensions keeping the number of z-like directions unity: this means that we stick to layered systems. Generalizing the problem further would raise both technical and physical questions not related to this work. In the following $d^d x$ stands for $dz\, d^{dy}y\, d^{dx}x$ and $d^d q$ for $dq_z\, d^{dy}q_y\, d^{dx}q_x$, $d = 1 + d_y + d_x$. The distinction between the x and y directions is again clear on the expression of the propagator along the maximum susceptibility line:

$$G^{-1}(q) = B_z q_z^2 + B_y q_y^2 + \cdots , \tag{16}$$

where the ellipsis represents fourth-order terms.

Before proceeding further we need some basic definitions; they follow the spirit of Ref. [10]. The main fluctuating variable, as we have defined the problem, is the layer displacement u. The partition function reads

$$Z(h) = \int D(u) \exp\left[-H(u) + \int d^d x\, h(\mathbf{x}) u(\mathbf{x}) \right] , \tag{17}$$

in which $\beta = 1$, $H(u) = H_1(u) + H_2(u)$, and $h(\mathbf{x})$ is the field conjugate to u. The free energy is then

$$F(h) = -\ln Z(h) , \tag{18}$$

and the $\langle u \rangle$ thermodynamic potential

$$\Gamma(\langle u \rangle) = F(h) + \int h \langle u \rangle d^d x , \tag{19}$$

from which one can extract vertex functions

$$\Gamma^n(\langle u(\mathbf{x}_1) \rangle \cdots \langle u(\mathbf{x}_n) \rangle) = \frac{\partial^n \Gamma}{\partial \langle u(\mathbf{x}_1) \rangle \cdots \partial \langle u(\mathbf{x}_n) \rangle} . \tag{20}$$

Since H depends on gradients of u only, only those fields h which do not involve a net force or torque over the system volume can lead to thermodynamic equilibrium. One is thus led to the choice

$$h = \nabla \cdot \mathbf{k} \tag{21}$$

with $k_z \neq 0$, $k_x = k_y = 0$ and

$$\Gamma(\langle u \rangle) = F(k_z) + \mathbf{k} \cdot \langle \nabla u \rangle \tag{22}$$

with the corresponding vertex functions:

$$f^{(n)}_{i_1 \cdots i_n} = \frac{1}{V} \frac{\partial^n \Gamma}{\partial M_{i_1} \cdots \partial M_{i_n}} , \tag{23}$$

where $M_{i_1} = \langle \partial_i u(\mathbf{x}_1) \rangle$ with, for vanishing wave vectors,

$$\begin{aligned} \Gamma^2(q) &= \sum_{i,j} f^{(2)}_{ij} q_i q_j + O(q^4) , \\ \Gamma^{(3)}(q_1, q_2, -q_1 - q_2) &= \sum_{i,j,k} f^{(3)}_{ijk} q_{i_1} q_{j_2} (-q_{k_1} - q_{k_2}) . \end{aligned} \tag{24}$$

The following identity, similar to the one holding in the Sm-A–Sm-A case (for position independent M_x and M_z) holds [34]:

$$M_x f^{(1)}_z = (M_z + 1) f^{(1)}_x . \tag{25}$$

Equation (25) allows us to generate a set of Ward identities linking the $f^{(n)}$ by successive derivations with respect to M_x and M_z. For instance, a proper choice of the coordinate axis ensures $M_x = 0$ and shows that the following lowest-order vertices are nonzero: $f^{(2)}_{zz}, f^{(2)}_{yy}, f^{(2)}_{xx}, f^{(3)}_{zzz}, f^{(3)}_{zxx}, f^{(3)}_{zyy}$. By construction the equilibrium state is given by $f^{(1)}_z = 0$, $f^{(2)}_{zz}$ describes the layer compression modulus and must vanish at the critical point just as the susceptibility usually diverges.

Whenever analytic expansions in terms of M_z have a meaning, the critical point is defined by

$$f^c_z = f^c_{zz} = f^c_{zzz} = 0 , \tag{26}$$

which implies, via the Ward identities,

$$f^c_{xx} = f^c_{zxx} = 0 . \tag{27}$$

These vertex functions can be calculated in perturbation

expansion. To one loop order

$$\Gamma(\langle u\rangle)=H(\langle u\rangle)+\tfrac{1}{2}\mathrm{tr}[\ln K(1,2)] \tag{28}$$

with

$$K(1,2)=\left[\frac{\partial H}{\partial u(\mathbf{x}_1)\partial u(\mathbf{x}_2)}\right]_{u=\langle u\rangle}, \tag{29}$$

that is, in q space,

$$\begin{aligned}K(q)=&[B_z+W_1M_z+(V_1/2)M_z^2+(V_{12}/2)M_x^2+(V'_{12}/2)M_y^2]q_z^2\\&+[B_x+W_2M_z+(V_2/2)M_x^2+(V_{12}/2)M_z^2+(V''_{12}/2)M_y^2]q_x^2\\&+[B_y+W'_2M_z+(V'_2/2)M_y^2+(V'_{12}/2)M_z^2+(V''_{12}/2)M_x^2]q_y^2+2(W_2M_x+V_{12}M_zM_x)q_xq_z\\&+2(W'_2M_y+V'_{12}M_zM_y)q_yq_z+K_{zz}q_z^4+K_{xx}q_x^4+K_{yy}q_y^4+K_{zy}q_z^2q_y^2+K_{xy}q_y^2q_x^2+K_{zx}q_z^2q_x^2\end{aligned} \tag{30}$$

and

$$\begin{aligned}\Gamma(\mathbf{M})=&hM_z+(B_z/2)M_z^2+(B_x/2)M_x^2+(B_y/2)M_y^2+(W_1/3!)M_z^3+(W_2/2)M_zM_x^2+(W'_2/2)M_zM_y^2\\&+(V_1/4!)M_z^4+(V_2/4!)M_x^4+(V'_2/4!)M_y^4+(V_{12}/4)M_z^2M_x^2+(V'_{12}/4)M_z^2M_y^2+(V''_{12}/4)M_x^2M_y^2\\&+\tfrac{1}{2}\,\mathrm{tr}\ln K(\mathbf{M},q)\end{aligned} \tag{31}$$

from which we extract by suitable differentiation.

$$\begin{aligned}&f_z(\mathbf{M}=\mathbf{0})=h+\tfrac{1}{2}\int_q(\partial K/\partial M_z)/K(\mathbf{M}=0)\,,\\&f_{zz}(\mathbf{M}=\mathbf{0})=B_1+\tfrac{1}{2}\int_q[(\partial^2K/\partial M_z^2)/K(\mathbf{M}=0)-(\delta K/\partial M_z)^2/K^2(M=0)]\,,\\&f_{zzz}(\mathbf{M}=\mathbf{0})=W_1+\tfrac{1}{2}\int_q[(\partial^3K/\partial M_z^3)/K(\mathbf{M}=0)-3(\partial K/\partial M_z)(\partial^2K/\partial M_z^2)/K^2(M=0)+2(\partial K/\partial M_z)^3/K^3(M=0)]\,,\\&f_{xx}(\mathbf{M}=\mathbf{0})=B_2+\tfrac{1}{2}\int_q[(\partial^2K/\partial M_x^2)/K(\mathbf{M}=0)-(\partial K/\partial M_x)^2/K^2(M=0)]\,,\\&f_{zxx}(\mathbf{M}=\mathbf{0})=W_2+\tfrac{1}{2}\int_{dq}[(\partial^3K/\partial M_z\partial M_x^2)/K(\mathbf{M}=0)-(\partial K/\partial M_z)(\partial^2K/\partial M_x^2)/K^2(M=0)\\&\qquad-2(\partial K/\partial M_x)(\partial^2K/\partial M_x\partial M_z)/K^2(M=0)+2(\partial K/\partial M_x)^2(\partial K/\partial M_z)/K^3(M=0)]\,.\end{aligned} \tag{32}$$

Note that in order to be consistent with rotational symmetry one has to keep terms like $(15W_1+10V_1)(\partial_z u)^5/5!$, $(3W_1+V_1)(\partial_z)^3(\partial_x u)^2/12$, and $(\partial_z u)^4/4$ in H for the calculation of f_{zzz} and f_{zxx}. In principle one should also keep nonlinear contributions coming from the covariant expression of the K_1 curvature terms. However, it is sufficient to consider isotropic second-order moduli in the (x,z) plane to assess the upper critical dimension (i.e., $K_{xx}=K_{yy}=K_{xz}/2$). Indeed, in this case, the Hamiltonian clearly exhibits the required symmetry and the Wards identities are automatically satisfied. The critical point is then defined by the conditions (26), which provide three equations for h_c, B_{1c}, and W_{1c} in terms of the other coupling constants. They also determine a_c, T_c, and E_c (or a_c, T_c, and P_c in the Sm-A–Sm-A critical point) as a function of the bare material parameters. They key remark is that, whereas $W^c=0$ is a solution in the liquid-vapor case, it is not here. Indeed, relations (26) reads

$$0=h_c+\tfrac{1}{2}\int d^dq\; W_c/K_c\,, \tag{33a}$$

$$0=B_{zc}-\tfrac{1}{2}\int d^dq\; W_c^2/K_c+\tfrac{1}{2}\int d^dq\; V_c/K_c\,, \tag{33b}$$

$$0=W_{1c}+\tfrac{1}{2}\int d^dq\; W_c^3/K_c^3-\tfrac{3}{2}\int d^dq\; W_cV_c/K_c^2+\tfrac{1}{4}\int d^dq\;\bar V_c/K_c\,, \tag{33c}$$

with

$$\begin{aligned}W_c&=W_{1c}q_z^2+W_{2c}q_x^2+W'_{2c}q_y^2\,,\\V_c&=V_{1c}q_z^2+V_{12c}q_x^2+V'_{12c}q_y^2\,,\\\bar V_c&=(15W_{1c}+10V_{1c})q_z^2+(3W_{1c}+V_{1c})q_x^2\,,\\K_c&=K_c(\mathbf{M}=\mathbf{0})\,.\end{aligned} \tag{34}$$

The main difference between relations (33) and the equation valid for the liquid-vapor case is the term $\frac{1}{4}\int d^dq\;\bar V_c/K_c$, which is required by rotational invariance here (and determined by the third- and fourth-order coupling constants W_1 and V_1), but can be canceled by a simple variable change in the liquid-vapor case. As a result W's are not all zero.

The upper critical dimension may now be calculated by testing the convergence of the high-temperature perturbation expansions. Let us, for instance, express the effective compressional elastic modulus $\bar B_z=f_{zz}(\mathbf{M}=\mathbf{0})$:

$$\begin{aligned}\bar B_z&=f_{zz}(\mathbf{M}=\mathbf{0})-f_{zz}^c(\mathbf{M}=\mathbf{0})\\&=B_z-B_{zc}+\tfrac{1}{2}\int d^dq\; W_c^2\left[\frac{1}{K_c^2}-\frac{1}{K^2}\right]\\&\quad+\tfrac{1}{2}\int d^dq\left[\frac{W_c^2}{K^2}-\frac{W^2}{K^2}\right]+\tfrac{1}{2}\int d^dq\; V_c\left[\frac{1}{K}-\frac{1}{K_c}\right]\end{aligned} \tag{35}$$

in which $K=\bar{B}_z q_z^2+B_y q_y^2+K_1 q_1^4$ is taken along the isochore and $K_c=K(\bar{B}_z=0)$, $W=W_c$. Integrating first over the y direction one can show

$$B_z-B_{zc}=\bar{B}_z\left[1+\frac{W_c^2}{2}\bar{B}_z^{(d+dy-6)/2}I(2)+\frac{V}{2}\bar{B}_z^{(d+dy-4)/2}I(1)+\cdots\right] \quad (36)$$

in which W_c and V are linear combinations of the W_1^c and V_1, respectively, and

$$I(n)\propto\int dx\, x^{d-1}\left[\frac{1}{(1+Kx^2)^{n-dy/2}}-\frac{1}{(Kx^2)^{n-dy/2}}\right]. \quad (37)$$

For $d+d_y>6$, $I(2)$ diverges like $\bar{B}_z^{-(d+dy-6)/2}$; this divergence is canceled by the $B_z^{(d+dy-6)/2}$ factor, and Eq. (36) provides a finite correction proportional to W_c^2 to the compressional elastic modulus. For $4<d+d_y<6$, $I(2)$ converges and the W_c^2 correction diverges like $\bar{B}_z^{(d+dy-6)/2}$ as $\bar{B}_z\to 0$ (note that the correction proportional to V is finite).

The borderline $d_c+d_{yc}=1+d_{xc}+2d_{yc}=6$ defines the upper critical dimensions. Since d_x and d_y may be arbitrarily chosen, one obtains a continuum of upper critical dimensions (Fig. 3) (note that the upper critical dimension of the fourth-order coupling constant is given by $1+d_{xc}+2d_{yc}=4$).

The pathologies of the higher vertex functions pointed out for the Sm-A–Sm-A case (10) hold more generally as long as there is a nonvanishing x-like direction. In particular one can expand the third-order vertex function as

$$f_{zzz}=W_1-W_{1c}+\tfrac{1}{2}W_c^2 B_z^{(d+dy-6)/2}I(3)+\cdots. \quad (38)$$

As expected for dimensions smaller than the upper critical one, the correction diverges as $B_z\to 0$. For $8>d+d_y>6$, $I(3)$ converges so that the correction to f_{zzz} converges, but in a nonanalytic way: like in the Sm-A–Sm-A case this is probably the signature of a non-mean-field coexistence curve. Only for $d+d_y>8$ is the correction linear in B_z [i.e., $I(3)$ diverges like $B_z^{-(d+dy-8)/2}$ and the mean-field is recovered]. Note also that the singularities of the higher vertex functions, pointed out in Ref. [10], also exist in this case as long as $d_x>0$, with critical dimensions $d+d_y=2n-1$, where n is the index of the vertex function.

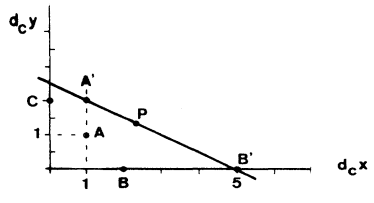

FIG. 3. Line of upper critical dimensions: Above this line problems are mean field, below they are critical. From any point P on this line, one can define an expansion in terms of $\epsilon_x=d_{xc}-d_x$ and $\epsilon_y=d_{yc}-d_y$. A corresponds to the Sm-A–Sm-A and Sm-C–Sm-I (Sm-C–Sm-F) critical points. B corresponds to the Sm-C^*–Sm-C^* and the Sm-A–Sm-A in transverse magnetic-field (plus negative magnetic anisotropy) critical points. C, which corresponds to a Sm-A–Sm-A critical point in a longitudinal magnetic field (plus positive magnetic anisotropy), cannot be obtained within this expansion scheme as explained in Ref. [30].

IV. RECURSION RELATIONS AND FIXED POINTS

With proper rescaling of the coordinate units and of u, and keeping only those terms of H which are relevant in the vicinity of the upper critical dimension, we start with the following Hamiltonian:

$$H(u)=\int d^d x\left[h\partial_z u+\tfrac{1}{2}B_z(\partial_z u)^2+\tfrac{1}{2}B_x(\partial_x u)^2+\tfrac{1}{2}(\partial_y^2 u)^2+\tfrac{1}{2}(1-C)(\partial_z^2 u)^2+(\partial_z\partial_x u)^2+\frac{W_1}{3!}(\partial_z u)^3+\frac{W_2}{2}\partial_x u(\partial_x u)^2\right]. \quad (39)$$

$H(u)$ differs from the Hamiltonian used in (10), in the presence of the $\frac{1}{2}(\partial_y u)^2$ term, and in the irrelevance of higher derivative or higher power, involving the y coordinate. The monumentum shell recursion relation may be developed using an anisotropic rescaling:

$$q_x'=bq_x,\quad q_y'=b^{2+\mu y}q_y,\quad q_z'=b^{1-\mu z}q_z,$$
$$u(b^{-1-\mu z}q_z,b^{-2-\mu y}q_y,b^{-1}q_x)=\xi^2 u(\mathbf{q}), \quad (40)$$
$$\xi^2=b^{d+\mu z+dy(\mu y+1)+dz\,\mu x-(2+\mu y)(2-\eta y)}.$$

This choice allows us to maintain $B_y=K_{xx}=K_{xz}=1$ (with obvious notations) during the renormalization procedure, and guarantees proper scaling of the uu correlation function at the critical point. Keeping $B_y=1$ expresses the finite value $P_c E_c$ of the modulus opposing tilt in the y direction at the critical point. The other choices $K_{xx}=K_{xz}=1$ are somewhat more arbitrary (one could have chosen, for instance, $K_{xz}=K_{zz}=1$ as well), but allow for a close comparison of the recursion relations with that of (10).

Each rescaling is followed by a partial momentum integration as usual; the integration volume may be either delimited by a unit hypersphere and a hyperellipsoid defined by $b^{-2}q_x^2+b^{-4-2\mu y}q_y^2+b^{-2-2\mu z}q_z^2=1$ with $b=e^{\delta l}$ or by a cylindrical volume (involving full integration in the y direction) of a cross section comprised between a unit "hypercircle" and a hyperellipsoid defined by $b^2q_x^2+b^{2+2\mu z}q_z^2=1$. We checked on particular examples that the two procedures yield the same results, but preferred the second one since it is computationally much less time consuming. We get

$$\frac{dh}{dl}=[d+2+d_y(\mu_y+1)-\eta_x-\mu_z/2]h+(W_1/2)I_{021}+(W_2/2)I_{021}, \quad (41)$$

$$\frac{dB_z}{dl}=(2-2\mu_z-\eta_x+4\mu_z-\eta_x\mu_x)B_z$$
$$-(W_1^2/2)I_{402}-(W_2^2/2)I_{042}-W_1W_2I_{222}\ , \quad (42)$$

$$\frac{dB_x}{dl}=[2-\eta_x(1+\mu_x)+2\mu_x]B_x-\frac{2}{d_x}W_2^2I_{222}\ , \quad (43)$$

$$\frac{dB_y}{dl}=[-2\mu_y-\eta_x(1+\mu_x)+4\mu_x]B_y\ , \quad (44)$$

$$\frac{dK_{xx}}{dl}=-\eta_x(1+\mu_x)K_1-R(1)W_2^2\ , \quad (45)$$

$$\frac{dK_{xz}}{dl}=[-\eta_x(1+\mu_x)-2\mu_z+2\mu_x]K_{xz}$$
$$-\frac{R(2)}{2}W_1^2-\frac{R(3)}{2}W_2^2-\frac{R(4)}{2}W_1W_2\ , \quad (46)$$

$$\frac{dK_{zz}}{dl}=[-\eta_x(1+\mu_x)-4\mu_z+4\mu_x]K_{zz}$$
$$-R(5)W_1^2-R(6)W_2^2-R(7)W_1W_2\ , \quad (47)$$

$$\frac{dW_1}{dl}=\left[\tfrac{1}{2}\epsilon-\tfrac{3}{2}\eta_x(1+\mu_x)-\tfrac{7}{2}\mu_z\right.$$
$$\left.-(d_c-\tfrac{19}{2})\mu_x+\left[\frac{d_c}{2}-3\right]\mu_y'\right]W_1$$
$$+R(8)W_1^3+R(11)W_2^3+R(9)W_1^2W_2$$
$$+R(10)W_2^2W_1\ , \quad (48)$$

$$\frac{dW_2}{dl}=\left[\tfrac{1}{2}\epsilon-\tfrac{3}{2}\eta_x(1+\mu_x)-\tfrac{3}{2}\mu_z\right.$$
$$\left.-(d_c-\tfrac{15}{2})\mu_x+\left[\frac{d_c}{2}-3\right]\mu_y'\right]W_2$$
$$+R(13)W_2^3+R(12)W_2^2W_1\ , \quad (49)$$

in which (in a first-order expansion scheme in $\epsilon_y=d_{yc}-d_y$ and $\epsilon_x=d_{xc}-d_x$)

$$I_{lmp}=\frac{Kd_x}{2\pi}\frac{\Gamma(p-dy/z)}{4\pi^{dy/2}\Gamma(p)}$$
$$\times\int_0^\pi d\theta\frac{(\cos\theta)^l(\sin\theta)^{d_{xc}-1+m}}{(1-c\cos^4\theta)^{p-d_{yc}/2}} \quad (50)$$

and the $R(i)$, given in Appendix B, are linear combinations of the I_{lmp}'s and as such, a function of C, d_x, and d_y.

The constraints on B_y, K_{xx}, and K_{xz} [i.e., Eqs. (44)–(46) set to zero] yield

$$\eta_x=-R(1)W_2^2\ ,$$
$$\mu_z-\mu_x=\frac{1}{2}\left[-\frac{R(2)}{2}W_1^2-\left[\frac{R(3)}{2}-R(1)\right]W_2^2\right.$$
$$\left.-\frac{R(4)}{2}W_1W_2\right]\ , \quad (51)$$
$$\mu_y'+2\mu_x=R(1)\frac{W_2^2}{2}\ ,$$

which allows us to reduce (41) to the following restricted set of equations:

$$\frac{dB_z}{dl}=2B_z-\frac{W_1^2}{2}I_{402}-\frac{W_2^2}{2}I_{042}-W_1W_2I_{222}\ , \quad (52)$$

$$\frac{dB_x}{dl}=2B_x-\frac{2}{d_x}W_2^2I_{222}\ , \quad (53)$$

$$\frac{dC}{dl}=W_1^2[R(5)-R(2)(1-C)]$$
$$+W_2^2[R(6)-(1-C)(R(3)-R(1))]$$
$$+W_1W_2[R(7)-R(4)(1-C)]\ , \quad (54)$$

$$\frac{dW_1}{dl}=\tfrac{1}{2}\epsilon W_1+W_1^3[R(8)+\tfrac{7}{8}R(2)]+W_2^3R(11)$$
$$+W_1^2W_2[R(9)+\tfrac{7}{8}R(4)]$$
$$+W_1W_2^2\left[R(10)+\tfrac{7}{8}R(3)+R(1)\left[\frac{d_c-7}{4}\right]\right]\ , \quad (55)$$

$$\frac{dW_2}{dl}=\tfrac{1}{2}\epsilon W_2+W_2^3\left[R(13)+\tfrac{3}{8}R(3)+\frac{d_c-3}{4}R(1)\right]$$
$$+W_2^2W_1[R(12)+\tfrac{3}{8}R(4)]+\tfrac{3}{8}R(2)W_2W_1^2\ . \quad (56)$$

Equations (52)–(56) have the same general structure as those written for the Sm-A–Sm-A case. They can thus be discussed in a similar framework. As expected, the Gaussian fixed point ($B_z=B_x=W_1=W_2=0$, C undetermined) is unstable for dimensions below the critical one. The expansion parameter is $\epsilon=\epsilon_x+2\epsilon_y$: although one can vary d_c and bring the point P in the (d_x,d_y) plane from where the expansion is performed to an apparently optimum location (i.e., minimizing the distance between that point and the physical one; see Fig. 3), the expansion parameter is independent of this choice. Indeed, a given physical problem defines d_y and d_x uniquely ($d_x=1,d_y=1$ in the Sm-C^*–Sm-C^* case; $d_x=2,d_y=0$ in the Sm-A–Sm-A one) hence $\epsilon=d_{xc}+2d_{yc}-d_x-2d_y=5-d_x-2d_y$, independent of the choice of P. What is then the virtue of changing the P location? In fact, varying the upper critical dimension changes the I_{lmp}'s and amounts to probing higher corrections in ϵ_x and ϵ_y (although in a somewhat uncontrolled fashion).

Aside from this remark, clearly B_z and B_x on the one hand, W_1 and W_2 on the other, are, respectively, of the order ϵ and $\epsilon^{1/2}$ at the fixed point, if it exists. Furthermore, the combination $W_2(dW_1/dl)-W_1(dW_2/dl)$ and dC/dl are homogeneous functions of the W's. Hence, at the fixed point they involve only the ratio (W_1/W_2) and C. The compatibility for these two equations provides the solution C^*, (W_1^*/W_2^*) from which all other quantities may be extracted. The solution can be obtained numerically; for a given d_c, one indeed finds a fixed point

$$W_1^* = x^*(\epsilon)^{1/2}(2\pi/Kd)_x^{1/2}(4\pi)^{d_y/4} ,$$

$$W_2^* = y^*(\epsilon)^{1/2}(2\pi/Kd_x)^{1/2}(4\pi)^{d_y/4} ,$$

$$B_z^* = \left[\frac{(x^*)^2}{4} J_{402} + \frac{(y^*)^2}{4} J_{042} + \frac{x^* y^*}{2} J_{222} \right] \epsilon , \tag{57}$$

$$B_x^* = \frac{(y^*)^2}{\alpha - 1} J_{222} \epsilon ;$$

$$\eta_x^* = -R(1)(y^*)^2 \epsilon ,$$

$$\mu_y^* = 1 - \eta_x/2 , \tag{58}$$

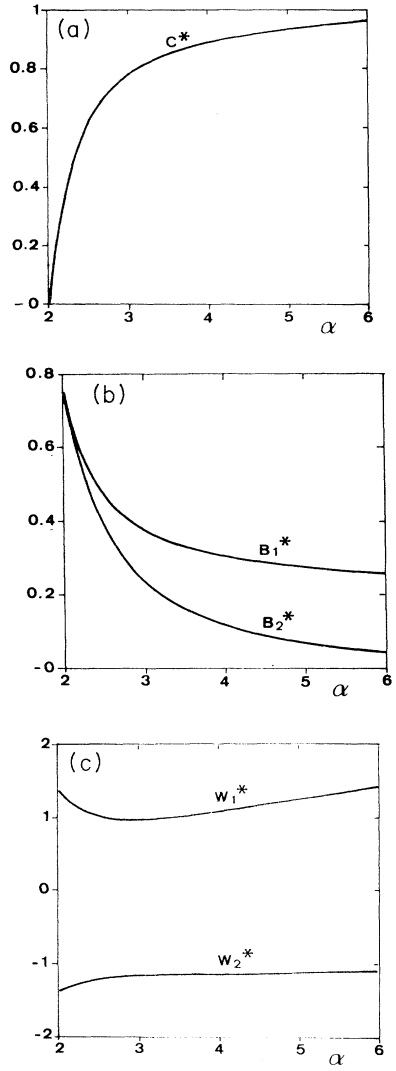

FIG. 4. Fixed points values (a) C^*, (b) B_z^* and B_x^*, and (c) W_1^* and W_2^* as a function of the parameter $\alpha = d_c - d_{cy} = 6 - 2d_{cy}$.

$$\mu_z^* = -\{ R(2)(x^*)^2 + R(4)(x^*)^2(y^*)^2 + [R(3) - 2R(1)](y^*)^2 \} \epsilon/4 ;$$

$$\eta_z = \eta_x + 4\mu_z = -\epsilon \{ R(2)(x^*)^2 + R(4)(x^*)^2(y^*)^2 + [R(3) + 4R(1)](y^*)^2 \}/4 , \tag{59}$$

$$\eta_y = 0 + O(\epsilon^2) ,$$

in which x^*, y^*, and η^* are numbers which depend only on d_c. It is possible to check for each of these fixed points that the electrostatic energy H_2 is irrelevant hence our claims of Sec. I. We have plotted in Fig. 4 the different fixed point values as a function of the parameter $\alpha = d_c - d_{cy} = 6 - 2d_{cy}$. The difference $B_z^* - B_x^*$ decreases continuously as α decreases, and vanishes for $\alpha = 2$. Right at this point $W_1^* = -W_2^*$, $C^* = 0$. Although this remarkable result may be checked directly in the fixed point equations, we have not been able to find a first-principles explanation for it. The isotropy of the harmonic term may look natural since in that case both the z and x directions are unidimensional and correspond to a vanishing elastic modulus, but the nonlinear asymmetry is more difficult to understand.

V. FIXED-POINT STABILITY AND CRITICAL EXPONENTS

As usual, the critical exponents are found by linearizing the recursions relations and finding the eigenvalues of the corresponding linear system of differential equations. Formally it can be written

$$\frac{dV_1}{dl} = \alpha_{ij} V_i , \tag{60}$$

with $V_1 = B_z(l) - B_z^*$, $V_2 = B_x(l) - B_x^*$, $V_3 = W_1(l) - W_1^*$, $V_4 = W_2(l) - W_2^*$, and $V_5 = C(l) - C^*$, and with the α_{ij}'s are calculated from the R_i's, with proper linearization. For obtaining the eigenvalues to first order in ϵ, α_{ij} may be considered as block diagonal:

$$\frac{d}{dl} V_i = 2\delta_{ij} V_j + \epsilon A_{ij} V_j , \quad i,j \in \{1,2\} \tag{61}$$

$$\frac{dV_i}{dl} = \epsilon A_{ij} V_j , \quad i,j \in \{3,4,5\} \tag{62}$$

with the redefinition $V_3 = \epsilon^{1/2} W_1$, $V_4 = \epsilon^{1/2} W_2$. The actual values of the A_{ij}'s are given in Appendix C. The largest eigenvalues, of order zero in ϵ, stem from Eq. (61):

$$\lambda_{1/2} = 2 + \frac{\epsilon}{2} \{ A_{11} + A_{22} \pm [4A_{12}A_{21} + (A_{11} - A_{22})^2]^{1/2} \} . \tag{63}$$

λ_1, by construction of the renormalization scheme, equals the inverse of the correlation length exponent ν_x. We

show in Appendix D that λ_2 is not an exponent linked to an external field but rather corresponds to the arbitrariness in the choice of M_z. More precisely, we show that M_z is the eigendirection corresponding to λ_2. Power counting requiring that $M_z(l)$ renormalizes as $l^{\omega}M_z(0)$ with

$$\omega=[d-2+dy(1+\mu_y)+3\mu_z+\eta_x]/2 \tag{64}$$

consistency of the renormalization-group scheme implies

$$\lambda_2\equiv\omega\ . \tag{65}$$

We show that this identity is formally satisfied by our recursion relations in Appendix D and checked directly its numerical relevance with a 10^{-6} accuracy. The remaining three eigenvalues are first order in ϵ, among which one is positive λ_3 and the other two are negative.

Up to now we have not considered the recursion relation (42) concerning the field h. Its structure implies the existence of a sixth eigenvalue

$$\lambda_h=[d+2+d_y(1+\mu_y)-\eta_x-\mu_z]/2\ , \tag{66}$$

the eigendirection of which corresponds essentially to h. We call t and g the eigendirections corresponding to the positive eigenvalues λ_1 and λ_3.

Note eventually that the renormalization of the coupling constant affecting the $q_xq_zu(q)u(-q)$ term defines a redundant operator for M_x, with

$$M_x(l)=e^{\omega' l}M_x(0) \tag{67}$$

and

$$\omega'=\tfrac{1}{2}[(d-2+dy(1+\mu_y)+\mu_z+\eta_x]\ . \tag{68}$$

Details are again given in Appendix D.

With relations (67), (68), (66), (64), (63), and (59), all physically important critical exponents can be calculated to first order in ϵ.

VI. DISCUSSION

The very existence of the above-discussed fixed point implies the following structure of the free energy in the critical domain:

$$F(t,g,h,M_z,M_x)=F_r+hM_z+e^{-[d+dy(1+\mu_y)+\mu_z]l}\times F_s(e^{\lambda_1 l}t,e^{\lambda_3 l}g,e^{\omega l}M_z,e^{\omega' l}M_x)\ , \tag{69}$$

in which F_r is the regular part of the free energy and F_s the singular part. *A priori* h and t are linear combinations of $(T-T_c)$ and $(E-E_c)$ [or $(P-P_c)$ and $(\mu-\mu_c)$ depending on the problem]. The g dependence on external fields is more subtle. The equation of state is obtained from Eq. (69) by differentiation with respect to M_z and setting $M_x=0$:

$$\frac{\partial F}{\partial M_z}(t,g,h,M_z,0)=0=h+e^{[-(d+dy(1+\mu_y)+\omega]l}f'_{sz}(e^{\lambda_1 l}t,e^{\lambda_3 l}g,e^{\omega l}M_z,0) \tag{70}$$

with

$$f'_{sz}(p,q,z,x)=\left.\frac{\partial F_s(p,q,z,x)}{\partial z}\right|_{p,q}\ ,$$

which can be rewritten

$$0=e^{\lambda_h l}h+f'_{sz}(e^{\lambda_1 l}t,e^{\lambda_z l}g,e^{\omega l}M_z,0)\ . \tag{71}$$

Fulfillment of Ward identities then requires

$$\frac{\partial^2 F}{\partial M_x^2}(t,g,h,M_z,0)=0=e^{[-(d+dy(1+\mu_y)+2\omega']l}\times f^{(2)}_{s_{xx}}(e^{\lambda_1 l}t,e^{\lambda_z l}g,e^{\omega l}M_z,0) \tag{72}$$

and

$$f^2_{sxx}(p,q,z,x)=\left.\frac{\partial^2 F_s}{\partial x}(p,q,z,x)\right|_{p,q,z}\ ,$$

which in turn imposes

$$e^{\lambda_z l}g=\varphi(e^{\lambda_1 l}t,e^{\omega l}M_z)\ . \tag{73}$$

For instance, if we approach the critical point on the critical isochore $M_z=0$, Eq. (73) implies

$$g=t^{\lambda_3/\lambda_1}\varphi(1,0)\ . \tag{74}$$

g does not obey the usual analyticity assumption on external fields. Indeed, the renormalization-group scheme used in this approach does not *a priori* satisfy rotational invariance, and only the restricted path obeying Eq. (73) does. As a result, the vanishing of g is determined by the vanishing of t and M_z.

Combining Eqs. (73) and (71) yields a more conventional equation of state:

$$M_z=e^{-\omega l}G(e^{\lambda_h l}h,e^{\lambda_1 l}t)\ . \tag{75}$$

On the one-phase side of the phase diagram G is univalued, whereas on the two-phase side two functions G^+ and G^- can be defined. It is important to understand that, because of the intrinsic asymmetry of the Hamiltonian (i.e., $\nabla_z u$ positive or negative correspond to fundamentally different situations even asymptotically at the critical point), coexistence is obtained for nonzero values of h.

This has two important consequences. (i) The law of rectilinear diameters is not obeyed; in the (h,t) plane the coexistence line follows the scaling from

$$h\propto t^{\Delta} \tag{76}$$

with $\Delta=\lambda_h/\lambda_1$. (ii) The order parameter measured along the coexistence curve obeys (since $\lambda_h\gg\lambda_1$ to first order in ϵ)

$$(M_z^+ - M_z^-) \propto h^{\omega/\lambda_h} G(1, \text{const}) , \tag{77}$$

or in terms of externally controlled parameters

$$M_z^+ - M_z^- \propto (T - T_c)^{1/\delta} , \tag{78}$$

with $\delta = \Delta/\beta = \lambda_h/\omega$. This result markedly contrasts with usual critical points, which are asymptotically symmetrical and for which $M_z^+ - M_z^- \propto (T - T_c)^{\beta}$. Clearly, the specific heat at zero stress (i.e., $\sigma_{zz} = 0$) is quite different from that taken at zero strain (i.e., $M_z = 0$):

$$C_p(M_z = 0) \propto \left. \frac{\partial^2 F}{\partial T^2} \right|_{M_z = 0} \propto t^{-\alpha} \tag{79}$$

with

$$\alpha = \{2 - [d + dy(1 + \mu_y) + \mu_z]/\lambda_1\} ,$$
$$C_p(\sigma_{zz} = 0) \propto \left. \frac{\partial^2 F}{\partial T^2} \right|_{\sigma_{zz} = 0} \propto h^{-\gamma/\Delta} (\text{or } t^{-\gamma} \text{ if } h = 0) \tag{80}$$

[the condition $\sigma_{zz} = 0$ is equivalent to Eq. (70)] with

$$\gamma = (2 - \eta_x - 2\mu_z)/\lambda_1 . \tag{81}$$

This difference is similar to the one between specific heat at constant volume and constant pressure for the liquid-vapor critical point. The existence of the fixed point also implies for the q-dependent nth-order vertex function along z

$$f_{sz\cdots z}^{(n)}(t, g, M_z, q_x, q_y, q_z) = b^{\{-[d + dy(1+\mu y) + \mu z] + n\omega\} l} f_{sz\cdots z}^{(n)}(e^{\lambda_1 l} t, e^{\lambda_1 l} g, e^{\omega l} M_z, e^{l} q_x, e^{(2+\mu_y) l} q_y, e^{(1+\mu_z) l} q_z) . \tag{82}$$

Note that the divergence of the perturbation theory for dimensions higher than d_c suggests that f_{sn} may not exist for $n \geq 3$. The two-point correlation function, which defines the experimental compressional elastic modulus B, obeys

$$f_{zz}^{(2)}(t, g, M_z, q_x, q_y, q_z) = e^{\{-[d + (\mu_y + 1) d_y + \mu_z] + 2\omega\} l} f_{zz}^{(2)}(e^{\lambda_1 l} t, e^{\lambda_z l} g, e^{\omega l} M_z, e^{l} q_x, e^{(2+\mu_y) l} q_y, e^{(1+\mu_z) l} q_z) , \tag{83}$$

or with the use of Eqs. (73) and (75)

$$f_{zz}^{(2)}(h, t, q_x, q_y, q_z) = e^{\{-[d + (\mu_y + 1) d_y + \mu_z] + 2\omega\} l} f(e^{\lambda_h l} h, e^{\lambda_1 l} t, e^{l} q_x, e^{(2+\mu_y) l} q_y, e^{(1+\mu_z) l} q_z) . \tag{84}$$

Identifying $f_{zz}^{(2)}(h, t, \mathbf{q} \simeq \mathbf{0})$ with the elastic modulus B yields

$$B \simeq h^{\gamma/\Delta} f(1, t h^{-1/\Delta}, \mathbf{q} = 0) . \tag{85}$$

In a noncontrolled approach of the critical point, both h and t are linear functions of $(T - T_c)$ and $(E - E_c)$, and the measured critical exponent will be γ/Δ. Only that path for which $h = 0$ leads to the exponent γ. Contrary to the case of asymptotically symmetrical critical points, the $h = 0$ line needs not be the isochore, but does correspond to the smallest B value line (largest susceptibility).

Expression (84) further implies the fully anisotropic correlation length

$$\xi_x \propto h^{-\nu_x/\Delta} \quad (t^{-\nu_x} \text{ if } h = 0), \nu_x = 1/\lambda_1$$
$$\xi_y \propto h^{-\nu_y/\Delta} \quad (t^{-\nu_y} \text{ if } h = 0), \ \nu_y = (2 + \mu_y)/\lambda_1$$
$$\xi_z \propto h^{-\nu_z/\Delta} \quad (t^{-\nu_z} \text{ if } h = 0), \ \nu_z = (1 + \mu_z)/\lambda_1$$

and divergences of the second-order elastic moduli, as announced in the Introduction.

We give in Tables I and II the expressions of the essential exponents to first order in ϵ, together with the coefficient of the first-order correction as a function of $\alpha = 6 - 2d$. Clearly, the resulting spread is too large for γ and β to allow for reliable predictions, except perhaps that γ is consistently smaller than 1. The corrections to the exponents divided by Δ are much smaller and lead to predictions which can be compared to experiment. For instance, we expect [34]

$$0.5 - 0.06\epsilon < \gamma/\Delta < 0.5 + 0.053\epsilon ,$$
$$0.5 - 0.039\epsilon < \beta/\Delta < 0.5 - 0.06\epsilon ,$$
$$2 - 0.0394\epsilon < \delta < 2 + 0.0624\epsilon ,$$
$$0.5 - 0.248\,859\,1\epsilon < \nu_x < 0.5 - 0.187\,239\,39\epsilon ,$$
$$1 - 0.594\,219\,12\epsilon < \nu_y < 1 - 0.038\,358\,49\epsilon ,$$
$$0.5 - 0.226\,708\,74\epsilon < \nu_z < 0.5 - 0.052\,286\,91\epsilon .$$

Comparison with experiment can be done for the Sm-A-Sm-A and the Sm-C^*-Sm-C^* critical points. In the first case, values of γ/Δ are obtained via compressional elastic modulus [31] and specific-heat measurements [32,33]. The latest values seem to converge towards (0.4 ± 0.08), which clearly excludes mean-field and Ising behavior, but is compatible with our estimate. In the second case γ, δ, and the order parameter exponent have been estimated [25]. Close to the critical point a regime with $\gamma < 1$, $\delta \simeq 2$ emerges which is quite compatible with our estimates, and excludes both mean-field and Ising behavior again. According to our remark that the coexistence is obtained for $h \neq 0$, the order parameter exponent is β/Δ rather than β, and the experimental value (although undistinguishable from mean field) is again compatible with our predictions.

Thus we believe that we have identified and characterized original universality classes which are relevant to layered systems. They are fundamentally anisotropic in that the exponents of the correlation lengths are different

TABLE I. Expressions of the essential exponents.

	General expression	First-order expression
ν_x	$1/\lambda_1$	$0.5-a+/4\epsilon$
ν_y	$(2+\mu y)/\lambda_1$	$1+(\mu' y/\epsilon-a+)/2\epsilon$
ν_z	$(1+\mu z)/\lambda_1$	$0.5+(2\mu z/\epsilon-a+)/4\epsilon$
β	ω/λ_1	$1-(1++a-)/2\epsilon$
Δ	λ_h/λ_1	$2+(ah-2a+)/2\epsilon$
γ	$(2-\eta x-2\mu z)/\lambda_1$	$1-(\eta x/\epsilon+\mu z/\epsilon+a+)/2\epsilon$
γ_{k1}	$\nu x \eta x$	$\eta x/2$
γ_{k2}	$\nu x(\eta x+4\mu z)$	$(\eta x/2+2\mu x)$
$\gamma k12$	$\nu x(\eta x+2\mu z)$	$(\eta x/2+\mu z)$
β/Δ	$\omega/\lambda h$	$0.5-(a-+ah/2)/4\epsilon$
γ/Δ	$(2-\eta x-2\mu z)/\lambda h$	$0.6-(2\mu z/\epsilon+\eta x/\epsilon+a+)/4\epsilon$
α	$2-[d+dy(\mu_y+1)+\mu z]/\lambda h$	$0.5-(1+3ah/2-\mu z/\epsilon -dy\mu' y/\epsilon)4\epsilon$

to first order in ϵ in three directions of space (only two in Sm-A–Sm-A case). Another important characteristic independent of the expansion scheme is that, at these points, the specific heat diverges like the susceptibility and that the field exponent δ is simply the inverse of the order parameter exponent (which is not β). The first set of experiments available is compatible with our calculations, which should be valid for at least four different physical situations: the Sm-A–Sm-A critical point, the Sm-C–Sm-I (Sm-C–Sm-F) critical point, the Sm-A–Sm-A critical point in a transverse field, and Sm-C^*–Sm-C^* critical point.

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APPENDIX A: ISOLATED CRITICAL POINT TERMINATING A Sm-C–Sm-I FIRST-ORDER LINE

We show in Figs. 5(a)–5(c) a few topologically possible phase diagrams involving Sm-C, Sm-I, Sm-A, and Sm-B_{hex} phases. Although two of them only involve an isolated critical point, we believe that in practice such critical points should occur fairly often in binary phase diagrams and that they control the physics of the "ghost transition" very often observed between the Sm-C and Sm-I phases. In particular the exponents related to the symmetry-breaking fields should certainly correspond to this fixed point (17). Since there is no symmetry change between Sm-C and Sm-I phases, they can only differ quantitatively: a natural choice for the description of the transition is obviously the intensity I_6 of the sixfold modulation of the x-ray scattering pattern in a λ scan (35). However, just as with the Sm-C^*–Sm-C^* transition, the choice of the layer displacement is better suited to take account of rotational symmetry. Indeed if u is defined with respect to the critical layer thickness at point c of Fig. 5(a), one can write $I_6-I_{6c} \propto \nabla_z u$ (where I_{6c} is the sixfold modulation at c). In fact, since the system always exhibits a twofold modulation as well close to the critical point, one could write also $I_2-I_{2c} \propto (\nabla_z u)$. Thus the similarity with the Sm-A–Sm-A or Sm-C^*–Sm-C^* critical points is fairly transparent. The only question comes from the coupling with the director field, which can be ignored altogether in the Sm-C^*–Sm-C^* case since it has a mass term, but needs to be considered more carefully here.

If we choose the average c director to point in the x direction, the nonlinear couplings with the displacement field can be written

TABLE II. First-order correction coefficients (to be multiplied by ϵ).

α	β	γ	β/Δ	γ/Δ
6	−0.412 900	−0.662 596	+0.062 413	−0.062 39
5.7	−0.424 459	−0.666 246 78		
5.4	−0.436 831	−0.670 037 36	+0.057 437 81	−0.059 165 39
5.1	−0.450 234 73	−0.673 961 49		
4.8	−0.464 822 8	−0.678 017 36		
4.5	−0.480 809 6	−0.682 210 68	+0.047 108 24	−0.053 592 31
3.64	−0.537 356 43	−0.695 286 01	+0.031 245 14	−0.047 719 65
3.28	−0.567 905 2	−0.701 580 25		
2.92	−0.604 819 40	−0.709 063 70	+0.008 007 17	−0.044 114 98
2.56	−0.650 770 37	−0.719 028 92	−0.010 879 2	0.045 00
2.2	−0.709 667 67	−0.735 020 75	−0.039 499 61	0.052 176 15

$$\delta F_{cu}=\int d^3x[\,W_{c1}\nabla_{xy}^2u(\nabla_{xx}^2u\,\nabla_{yy}^2u)C_y+W_{c2}\nabla_z u(\nabla_x C_y)^2 + W_{c2}\nabla_z u(\nabla_x C_y)^2+W_{c4}\nabla_z u(\nabla_z C_y)^2 +u_{c1}(\nabla_{xy}^2u)^2C_y^2+u_{c2}\nabla_{xy}^2u(\nabla_{xx}^2u\,\nabla_{yy}^2u)C_y^2 +u_{c3}(\nabla_z u)^2(\nabla_x C_y)^2+\cdots\,]\,, \tag{A1}$$

where the ellipsis represents similar fourth-order terms. With the rescaling of the Sm-A–Sm-A problem

$$u(b^{-1-\mu_z}k_z,b^{-1}k_\perp)=\xi^2u(\mathbf{k}) \qquad \xi^2=b^{d+4+\mu_z-\eta}\,, \tag{A2}$$

and ensuring the C_y director fluctuations to be marginal with

$$C_y(b^{-1-\mu_z}k_z,b^{-1}k_\perp)=\xi_c^2C_y(\mathbf{k})\,, \qquad \xi_c=b^{d+2+\mu_z-\eta_c}\,, \tag{A3}$$

it is straightforward to show that any generalization of δF_{cu} to higher dimension is irrelevant near d_c. This is sufficient to ascertain that the Sm-C–Sm-I, Sm-C–Sm-F critical points should belong to the Sm-A–Sm-A universality class (note that even in our more general scaling used throughout this paper, δF_{cu} can be shown to be irrelevant, with the choice $\xi_c=b^{d+d_yu_y+\mu_z-\eta_c}$).

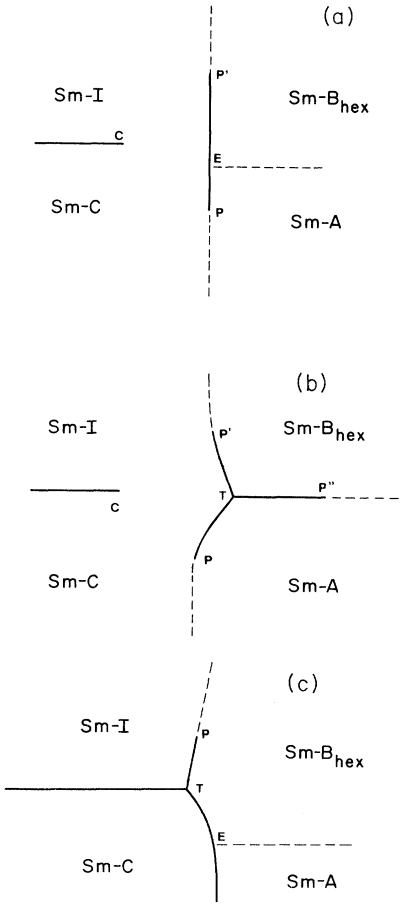

FIG. 5. Examples of phase diagrams compatible with thermodynamics, and involving Sm-C, Sm-I (or Sm-F), Sm-B_{hex}, and Sm-A phases in a (μ,T) plane. Solid lines represent first-order and dotted lines second-order transitions. Note that the Sm-C–Sm-I transition should always be first order. Ghost Sm-C–Sm-I transitions often quoted to replace a regular Sm-C–Sm-I one, should correspond to supercritical fluctuations in the vicinity of the isolated critical point C. (a) Case involving one isolated critical point C, called Sm-C–Sm-I critical point in the text, one critical end point E, and two tricritical points P,P'. (b) Case involving one isolated critical point C, one ordinary triple point T, and three tricritical points P,P',P''. (c) Alternative topologies involving one critical end point E, one triple point T, and one tricritical point P.

APPENDIX B: R_1's COEFFICIENTS APPEARING IN EQS. (42)–(50)

$$R(1)=-\frac{8}{d_x}(I_{423}+I_{243})-\frac{24}{d_x(d_x+2)}I_{243} +\frac{96}{d_x(d_x+2)}(I_{644}+2I_{464}+I_{284})\,,$$

$$R(2)=-I_{423}-I_{603}-\frac{2}{d_x}I_{423} +\frac{8}{d_x}(I_{824}+2I_{644}+I_{464})\,,$$

$$R(3)=\frac{7}{2d_x}I_{022}-\left[\frac{14}{d_x}+1\right]I_{063}-\left[\frac{52}{d_x}+1\right]I_{243} -\frac{32}{d_x}(1-c)I_{423}+\frac{32}{d_x}(1-c)^2I_{824} +128\frac{(1-c)}{d_x}I_{644}+\frac{168-64C}{d_x}I_{464} +\frac{80}{d_x}I_{284}+\frac{8}{d_x}I_{0104}\,,$$

$$R(4)=-2I_{423}-2I_{243}-\frac{36}{d_x}I_{423}-\frac{24}{d_x}I_{243} +\frac{144-64C}{d_x}I_{644}+\frac{96}{d_x}I_{464}+\frac{16}{d_x}I_{284} +\frac{64(1-C)}{d_x}I_{824}\,,$$

$$R(5)=\tfrac{1}{2}I_{202}-7(1-C)I_{603}-5I_{423} +8(1-C)^2I_{1004}+16(1-C)I_{824}+8I_{644}\,,$$

$$R(6)=-3(1-C)I_{243}-I_{063}+8(1-C)^2I_{644} +16(1-C)I_{464}+8I_{284}\,,$$

$$R(7)=-10(1-C)I_{423}-6I_{243}+16(1-C)^2I_{824} +32(1-C)I_{644}+16I_{464}\,,$$

$$R(8)=I_{603},\quad R(9)=3I_{423},\quad R(10)=3I_{243}\,,$$

$$R(11)=I_{063},\quad R(12)=\frac{4}{d_x}I_{423},\quad R(13)=\frac{4}{d_x}I_{243}\,.$$

APPENDIX C: MATRIX ELEMENTS A_{ij} IN EQ. (60)

$$
\begin{aligned}
A_{11} &= X^{*2}R(2)/2 + Y^{*2}R(3)/2 + X^*Y^*R(4)/2 \\
&\quad - X^{*2}S'_{B1}(11) - Y^{*2}S'_{B1}(12) - X^*Y^*S'_{B1}(13) , \\
A_{33} &= 0.5 + 3X^{*2}S(1) + 2X^*Y^*S(2) + Y^{*2}S(3) , \\
A_{34} &= X^{*2}S(2) + 2X^*Y^*S(3) + 3Y^{*2}S(4) , \\
A_{43} &= 2X^*Y^*S(5) + Y^{*2}S(6) , \\
A_{44} &= 0.5 + X^{*2}S(5) + 2X^*Y^*S(6) + 3Y^{*2}S(7) , \\
A_{56} &= X^{*2}S'_c(8) + X^*Y^*S'_c(9) + Y^{*2}S'_c(10) , \\
A_{35} &= X^{*3}S'_c(1) + X^{*2}Y^*S'_c(2) \\
&\quad + X^*Y^{*2}S'_c(3) + Y^{*3}S'_c(4) , \\
A_{45} &= X^{*2}Y^*S'_c(5) + X^*Y^{*2}S'_c(6) + Y^{*3}S'_c(7) , \\
A_{53} &= 2X^*S(8) + Y^*S(9) , \\
A_{54} &= X^*S(9) + 2Y^*S(10) , \\
A_{12} &= X^{*2}I_{423} + 2X^*Y^*I_{243} + Y^{*2}I_{063} , \\
A_{21} &= 4/(\alpha-1)Y^{*2}I_{423} , \\
A_{22} &= [R(1) + 4/(\alpha-1)I_{243}]Y^{*2} .
\end{aligned}
$$

APPENDIX D: REDUNDANT OPERATORS

In our derivation of recursion relations (42)–(50) no explicit reference to the actual state with respect to which the layer displacement u is defined has been made. This implies that if the recursion relations are valid for an expansion centered on the critical point, they should be valid as well for an expansion from a point slightly off. However, it is easy to express the elastic moduli B_x and B_z defined from one of these states, in terms of the other (at the lowest relevant order):

$$
\begin{aligned}
\bar{B}_z &= B_z - W_1 M_z , \\
\bar{B}_x &= B_x - W_2 M_z ,
\end{aligned}
\tag{D1}
$$

$$
\begin{aligned}
\nabla_z u &= -M_z + \nabla_z u_c , \\
\nabla_x u &= -M_x + \nabla_x u_c ,
\end{aligned}
\tag{D2}
$$

in which u_c is defined with respect to the critical state and u with respect to a state of averaged compression M_z and tilt M_x (themselves defined with respect to the critical state). $\bar{B}_x$ and $\bar{B}_z$ are the effective coupling constants in the u representation and B_x and B_z in the u_c one. Close enough to criticality (D1) implies the validity of

$$
\begin{aligned}
\delta\bar{B}_z &= \delta B_z - W_1^* M_z , \\
\delta\bar{B}_x &= \delta B_x - W_2^* M_z .
\end{aligned}
\tag{D3}
$$

Since one expects the scaling law $M_z(l) = e^{\omega l} M_z(0)$ to be valid, the consistence of (D3) with the renormalization-group scheme requires ω to be identical to one of the eigenvalues λ_1 or λ_2, and the vector V_M of components (W_1, W_2) in the $(\delta B_x, \delta B_z)$ plane to be parallel to the corresponding eigenvector. In the following, we show $\omega = \lambda_2$ and $V_M \| V_2$ (eigenvector of λ_2). The proof of the equality $\omega = \lambda_2$ follows that given in [10], and details are given in [34].

The V_2 direction can easily be extracted from Eq. (61):

$$
V_2 \propto (-\epsilon A_{12}, 2 + \epsilon A_{11} - \lambda_2) \propto (2 + \epsilon A_{22} - \lambda_2, -A_{21})
$$

Thus $V_2 \| V_M$ requires the equality

$$
\frac{W_1^*}{-\epsilon A_{12}} = \frac{W_2^*}{2 + \epsilon A_{11} - \lambda_2} \tag{D4}
$$

or, using the expression (63) of λ_2,

$$
A_{11} - A_{22} + [(A_{11} - A_{22})^2 + 4A_{12}A_{21}]^{1/2} = -2\frac{W_2^*}{W_1^*} A_{12} . \tag{D5}
$$

As in (10) we can show

$$
\begin{aligned}
2 + \epsilon A_{11} - \omega &= -\epsilon \frac{W_2^*}{W_1^*} A_{12} , \\
2 + \epsilon A_{22} - \omega &= -\epsilon \frac{W_1^*}{W_2^*} A_{21} ,
\end{aligned}
\tag{D6}
$$

from which (D5) is obvious (note that $W_2^*/W_1^* < 0$). Hence M_z is an eigendirection of the renormalization-group equations, with eigenvalue ω, hence its use in the scaling relation (69).

Similarly a nonzero M_x generates a term $-W_2 M_x \nabla_x u \nabla_z u$ in the Hamiltonian, which implies the more general existence of a B_{xz} coupling constant the renormalization of which reads

$$
\frac{dB_{xz}}{dl} = (2 - \mu_z - \eta_x) B_{xz} - \frac{d_x^{1/2}}{2} \int \frac{WW'}{K^2} , \tag{D7}
$$

with $W = W_1 \cos^2\theta + W_2 \sin^2\theta$, $W' = 2W_2 \cos\theta \sin\theta / d_x^{1/2}$, and $K = B_z \cos^2\theta + B_x \sin^2\theta + B_{xz} \sin\theta \cos\theta + 1 - C \cos^4\theta$. The integration bears as before on the θ variable. Again consistency requires M_x to be the eigendirection and ω' the eigenvalue. Linearizing (D7) in the vicinity of the fixed point shows that this equation is decoupled from the others by symmetry, which implies that M_x is indeed the eigendirection. The corresponding eigenvalue reads

$$
\lambda_{xz} = 2 - \eta_x - \mu_z + 2d_x^{1/2} \int WW' \sin\theta \cos\theta / K^3 . \tag{D8}
$$

Hence

$$
\begin{aligned}
\lambda_{xz} - \omega' &= [\epsilon - 3\eta_x - 3\mu_x + (d_c - 6)\mu_y]/2 \\
&\quad + \int WW' \sin\theta \cos\theta / K^3 \lambda_{xz} - \omega' \\
&= \frac{1}{W_2} \left[\frac{dW_2}{dl} - \int \frac{WW'^2}{K^3} \right] \\
&\quad + \frac{2}{d_x^{1/2}} \int \frac{W'W}{K^3} \sin\theta \cos\theta .
\end{aligned}
\tag{D9}
$$

The vanishing of (D9) at the fixed point shows that indeed the eigenvalue is identical to ω'.

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